The punctual Hilbert schemes of degree two for monomial plane curve singularities†

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In this paper, we show that punctual Hilbert schemes of degree two for monomial plane curve singularities are isomorphic to a projective line.

Key words: Hilbert schemes of points, irreducible curve singularity

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero. In this paper, we consider a monomial plane curve singularity given by

(1) \[ x = t^a, \quad y = t^b \quad (a < b, \ \gcd(a, b) = 1). \]

Its local ring $\mathcal{O}$ is isomorphic to $k[[t^a, t^b]]$. Let $\delta$ be the $\delta$-invariant of $\mathcal{O}$. In our case, we have $\delta = (a - 1)(b - 1)/2$. In 6), Pfister and Steenbrink defined a special subset of the Grassmannian $\text{Gr}(\delta, \mathcal{O}/I(2\delta))$ for a given monomial plane curve singularity. We call it the Pfister-Steenbrink variety (PS variety) for the given singularity. They showed the existence of the punctual Hilbert scheme of degree $r$ which parametrizes the ideals of codimension $r$ in $\mathcal{O}$. It is realized as a connected component of the PS variety.

Pfister and Steenbrink analyzed the structure of the PS varieties for certain curve singularities. The first author argued the rationality of punctual Hilbert schemes for monomial plane curve singularities in 11). The punctual Hilbert schemes for monomial plane curve singularities of types $A_{2d}$, $E_6$ and $E_8$ were also studied in 9) and 10). On the other hand, the punctual Hilbert schemes for nodal singularity were considered in 7). The structure of the punctual Hilbert schemes for cuspidal and nodal singularities applied to the string theory in 5). Also refer to 1) and 3) about punctual Hilbert schemes. Our main theorem is stated as follows:

Theorem 1. Let $C$ be a monomial curve singularity given by the parametrization (1), the punctual Hilbert scheme of degree 2 for $C$ is isomorphic to a projective line.

In Section 2, we briefly recall Pfister-Steenbrink theory. In Section 3, we prove Theorem 1. Finally, we consider the case of the plane curve singularity defined by $x = t^4, y = t^5$ in Section 4.

2. Preliminaries

We recall Pfister-Steenbrink theory and prove some lemmas needed later. We fix notations. Consider a monomial plane curve singularity whose local ring is $\mathcal{O} = k[[t^a, t^b]]$. The notions explained in this section hold more general situations. For details, see 6). We denote by $\mathcal{O}$ the normalization of $\mathcal{O}$. Namely, $\mathcal{O} = k[[t]]$. We call the set $\Gamma := \{\text{ord}(f) | f \in \mathcal{O}\}$ the semigroup of $\mathcal{O}$. An element of the set $G := \mathbb{N} \setminus \Gamma$ is called a gap of $\Gamma$. We may assume that the semigroup $\Gamma$ of $\mathcal{O}$ is minimally generated by $a, b$. For $n \in \mathbb{N}$, set $\mathcal{T}(n) := \{\text{ord}(f) | f \in \mathcal{O}\}$ and $I(n) := \mathcal{T}(n) \cap \mathcal{O}$. We call $\gamma := \min\{n | I(n) \subset \mathcal{O}\}$ the conductor of $\Gamma$. Define the $\delta$-invariant of $\mathcal{O}$ to be $\dim_k(\mathcal{O}/\mathcal{O}) = \gamma G$. Then the relation $\delta + 1 \leq \gamma \leq 2\delta$ holds. Furthermore, the following lemma is known. See 4) and 8).

Lemma 2 (Gorenstein). For a plane curve singularity, we have $c = 2\delta$. 

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For $p, q \in \mathbb{N} (p < q)$, put $[p, q] := \{x \in \mathbb{N} | p \leq x \leq q \}$. For a nonzero ideal $I$ in $\mathcal{O}$, we call $r := r(I) := \text{dim} \mathcal{O}/I$ the codimension of $I$. We denote by $\Gamma(I) := \{ \text{ord}(f) | f \in I \}$ the order set of $I$. Set $G(I) := \Gamma \setminus \Gamma(I)$ and $c(I) := \max\{n \in \Gamma(I) | n - 1 \notin \Gamma(I) \}$.

**Definition 3.** An element $J$ of $\text{Gr} (\delta, \mathcal{O}/I(2\delta))$ is called good, if the multiplication $\mathcal{O} \times J \ni (f, \pi) \mapsto \overline{f} \pi \in J$ is defined and $J$ is an $\mathcal{O}$-submodule with respect to this multiplication. Set $M := \{ J \in \text{Gr} (\delta, \mathcal{O}/I(2\delta)) | J \text{ is good} \}$, $I_r := \{ I | I \text{ is an ideal of } \mathcal{O} \text{ with } \text{dim} \mathcal{O}/I = r \}$.

We prove some lemmas.

**Lemma 4.** An ideal $I$ of $\mathcal{O}$ belongs to $I_r$ if and only if we have $\mathcal{O}G(I) = r$.

**Proof.** We easily see that an ideal $I$ belongs to $I_r$ if and only if the relation
\[(2) \quad \mathcal{O}/I = \left\{ \sum_{i=0}^{r-1} a_i t^{d_i} | d_i \in G(I), d_i < d_{i+1} \right\} \]
holds.

Lemma 4 implies that the codimension of $I$ depends on $\Gamma(I)$. For a $\Gamma$-module $S$, we denote by $I(S)$ the set of all ideals in $\mathcal{O}$ with $\Gamma(I) = S$.

**Proposition 5.** There exists a finite number of distinct $\Gamma$-modules $S_1, \cdots, S_h$ such that
\[(3) \quad I_r = \bigcup_{i=1}^{h} I(S_i) \]
where $I(S_i) \cap I(S_j) = \emptyset$ for $i \neq j$.

**Proof.** For a given codimension $r$, there exists a finite number of sets of gaps which satisfy the condition (2). So there exists finitely many distinct $\Gamma$-modules $S_1, \cdots, S_h$ which are the order sets of ideals in $I_r$. We obtain the desired decomposition (3). It is clear that $I(S_i) \neq I(S_j)$ for $i \neq j$.

**Remark 6.** An algorithm to compute $\Gamma$-modules $S_1, \cdots, S_h$ was given in (12).

We consider the following composition map:
\[\psi : M \to \text{Gr}(\delta, 2\delta) \to M_{\delta, 2\delta}(k)/ \sim \to \mathbb{P}^N\]
Put $J = \langle f_1, \cdots, f_h \rangle_k$ where $f_i = \sum_{j=0}^{2\delta} a_{ij} t^j \in \mathcal{O}/I(2\delta)$. We identify $f_i$ with a point $a_i = \langle a_{i0}, \cdots, a_{i2\delta-1} \rangle_{2\delta}$ of $k^{2\delta}$. The first map in $\psi$ is given by this identification. Let $A_J$ be a $\delta \times 2\delta$ matrix whose $i$-th row is $a_i$. We call $A_J$ the representative matrix of $J$. The second map in $\psi$ sends a $k$-vector space $(a_1, \cdots, a_h)_{k}$ to the coset $\overline{A}_J$ of $A_J$. Here the equivalence relation $\sim$ is the similarity of matrices. We may assume that $\overline{A}_J$ is represented by the reduced row echelon form. The third map in $\psi$ is Pl"ucker embedding with $N = \binom{2\delta}{\delta} - 1$. For $r > 0$, Pfister and Steenbrink defined a map $\varphi_r : I_r \to M$ by $\varphi_r(I) = t^{-r}I(2\delta)$. For $\varphi_r$, the following proposition was shown in (i):

**Proposition 7.** The map $\varphi_r$ has the following properties.

(i) The map $\varphi_r$ is injective for any $r$.

(ii) The map $\varphi_r$ is bijective for $r \geq 2\delta$.

(iii) The set $\varphi_r(I_r)$ is a Zariski closed set in $M$.

**Definition 8.** For a given monomial plane curve singularity, we call $M$ and $M_r := \varphi_r(I_r)$ the Pfister-Steenbrink variety (PS variety) and the punctual Hilbert scheme of degree $r$ respectively.

The following fact follows from (ii) in Proposition 7.

**Corollary 9.** The punctual Hilbert scheme $M_r$ with $r \geq 2\delta$ coincides with the PS variety $M_r$.

We define Schubert cell $W_{a_1, \cdots, a_h}$ for $\delta \geq a_1 \geq \cdots \geq a_h \geq 0$ to be the set of all elements $W$ in $\text{Gr}(\delta, \mathcal{O}/I(2\delta))$ which satisfy $\text{dim}(W \cap V_{\delta+a_i}) = i$ for $1 \leq i \leq \delta$ and $\text{dim}(W \cap V_j) < i$ for $j < \delta + i - a_i$. The $2\delta$-dimensional $k$-vector space $\mathcal{O}/I(2\delta)$ has the canonical flag
\[0 \subset V_1 \subset V_2 \subset \cdots \subset V_{2\delta} = \mathcal{O}/I(2\delta)\]
where $V_i = \mathcal{T}(2\delta - i)/I(2\delta)$ for $1 \leq i \leq 2\delta$. This induces a partition of $\text{Gr}(\delta, \mathcal{O}/I(2\delta))$ into Schubert cells $W_{a_1, \cdots, a_h}$. For an index set $A = \{a_1, \cdots, a_h\}$, we sometimes write $W_A$ instead of $W_{a_1, \cdots, a_h}$.

**Proposition 10.** For the Schubert cells, we have $W_{b_1, \cdots, b_h} \subset W_{a_1, \cdots, a_h}$ if and only if $b_i \geq a_i$ holds for $1 \leq i \leq \delta$. 

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For the details about Schubert cells, see 2).

Let \( \Delta \) be a subset of \([0, 2\delta - 1]\) such that \( \sharp\Delta = \delta \) and \( \Delta \cup [2\delta, \infty) \) is a \( \Gamma \)-module. Then we define \( \mathcal{M}_e(\Delta) \) to be the subset of \( \mathcal{M}_e \) parametrizing ideals \( I \) with \( \Delta = (\Gamma(I) - r) \cap [0, 2\delta - 1] \).

**Lemma 11.** Put \( \Delta = \{b_1, \ldots, b_3\} \) where \( 0 \leq b_1 < \cdots < b_3 < 2\delta \). Setting
\[
(4) \quad a_{s-i+1} = b_i - i + 1 \quad 1 \leq i \leq \delta,
\]
we have \( \mathcal{M}_e(\Delta) = \mathcal{M}_e \cap W_{a_1, \ldots, a_3} \) for \( r \in \mathbb{N} \).

**Proof.** Under the same conditions, the relation \( \mathcal{M}_{2\delta}(\Delta) = \mathcal{M}_{2\delta} \cap W_{a_1, \ldots, a_3} \) holds (see Lemma 5 in 1)). So we have
\[
\mathcal{M}_e(\Delta) = \mathcal{M}_e(\Delta) \cap \mathcal{M}_{2\delta}(\Delta) = \mathcal{M}_e \cap (\mathcal{M}_{2\delta} \cap W_{a_1, \ldots, a_3}) = \mathcal{M}_e \cap W_{a_1, \ldots, a_3}.
\]

For a component \( \mathcal{I}(S_i) \) of the decomposition (3), we have \( \mathcal{M}_e(\Delta_i) = (\psi \circ \varphi_r)(\mathcal{I}(S_i)) \) where \( \Delta_i = (S_i - r) \cap [0, 2\delta - 1] \). The decompositions of \( \mathcal{M}_e \) follows from Proposition 5.

**Corollary 12.** Corresponding to (3), we have
\[
(5) \quad \mathcal{M}_e = \bigcup_{i=1}^{h} \mathcal{M}_e(\Delta_i).
\]

**Proposition 13.** Each component \( \mathcal{M}_e(\Delta_i) \) in the decomposition (5) is isomorphic to an affine space whose dimension equals the number of coefficients in the generators of \( \mathcal{M}_e(\Delta_i) \) (as \( \mathcal{O} \)-modules).

### 3. Proof of Theorem 1

In this section, we prove Theorem 1. Let \( C \) be a plane curve singularity given by (1). Put \( \mathcal{O} = k[[t^a, t^b]] \). It follows from Lemma 2 that \( 2\delta = c = (a - 1)(b - 1) \). Write
\[
\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{2\delta}, \ldots\}
\]
where \( \gamma_i < \gamma_{i+1} \), \( \gamma_1 = 0 \), \( \gamma_2 = a \), \( \gamma_{2\delta} = (a - 1)(b - 1) \) and \( \gamma_{2\delta+i} = (a - 1)(b - 1) + i \) for any \( i \in \mathbb{N} \). By Lemma 4, we see that an ideal \( I \) of \( \mathcal{O} \) belongs to \( \mathcal{I}_2 \) iff \( \Gamma(I) \) coincides with one of \( \Gamma \)-modules \( S_1 = (2a, b) \) and \( S_2 = (a, b+a) \). Here the notation \( (p, q) \) assigns the \( \Gamma \)-module generated by \( p \) and \( q \). In our case, they are described as
\[
S_1 = \{\gamma_3, \gamma_4, \gamma_5, \gamma_6, \ldots\} \quad (\gamma_3 = 2a),
S_2 = \{\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots\} \quad (\gamma_1 = a, \gamma_j = b).
\]
So the decomposition (3) in Proposition 5 is determined as
\[
(6) \quad \mathcal{I}_2 = \mathcal{I}(S_1) \cup \mathcal{I}(S_2).
\]
Furthermore, we easily see that
\[
\mathcal{I}(S_1) = \{(t^{2n}, t^b)\},
\]
\[
\mathcal{I}(S_2) = \{(t^n + pt^b, t^{b+a}) | p \in k\}.
\]
We obtain the decomposition (5) of \( \mathcal{M}_2 \) as
\[
(8) \quad \mathcal{M}_2 = \mathcal{M}_2(\Delta_1) \cup \mathcal{M}_2(\Delta_2)
\]
where
\[
\Delta_1 := (S_1 - 2) \cap [0, 2\delta - 1] = \{\gamma_3 - 2, \gamma_4 - 2, \ldots, \gamma_{2\delta+1} - 2, \gamma_{2\delta+2} - 2\},
\]
\[
\Delta_2 := (S_2 - 2) \cap [0, 2\delta - 1] = \{\gamma_2 - 2, \ldots, \gamma_{j-1} - 2, \gamma_{j+1} - 2, \ldots, \gamma_{2\delta+2} - 2\}.
\]

Here we rewrite \( \Delta_1 \) and \( \Delta_2 \) as
\[
\Delta_1 = \{m_1, \ldots, m_{2\delta}\} \quad (m_i < m_{i+1}),
\]
\[
\Delta_2 = \{n_1, \ldots, n_{2\delta}\} \quad (n_i < n_{i+1}).
\]

According to (4), we define index set \( \Lambda_1 = \{p_1, \ldots, p_{2\delta}\} \) (resp. \( \Lambda_2 = \{q_1, \ldots, q_{2\delta}\} \) by \( p_{i+2\delta} = m_i - i + 1 \) (resp. \( q_{i+2\delta} = n_i - i + 1 \)) for \( 1 \leq i \leq \delta \). It follows from (8), Lemma 11 and Proposition 10 that
\[
\mathcal{M}_2 = (\mathcal{M}_2 \cap W_{\Lambda_1}) \cup (\mathcal{M}_2 \cap W_{\Lambda_2}) = \mathcal{M}_2 \cap (W_{\Lambda_1} \cup W_{\Lambda_2}) = \mathcal{M}_2(\Delta_2).
\]
Hence we have \( \mathcal{M}_2 = W_2(\Delta_2) \). It follows from Proposition 13 and (7) that \( \mathcal{M}_2(\Delta_2) = A_k \).
Hence we conclude that \( \mathcal{M}_2 = \mathbb{P}^1 \). Theorem 1 has proved.

### 4. Example

In this section, we consider the plane curve singularity which given by \( x = t^4 \), \( y = t^5 \). We have \( \mathcal{O} = k[[t^4, t^5]] \) and
\[
\Gamma = \{0, 4, 5, 8, 9, 10, 12, 13, \cdots\}
\]
where \( c = 12 \). Putting
\[
S_1 = \{5, 8, 9, 10, 12, 13, \cdots\},
S_2 = \{4, 8, 9, 10, 12, 13, \cdots\},
\]
we obtain the decomposition (6) of \( I_2 \) and have
\[
I(S_1) = \{(t^5, t^8)\}, \\
I(S_2) = \{(t^4 + pt^3, t^9) | p \in k\}.
\]
The images of these set by \( \varphi_r \) are
\[
\mathcal{M}(\Delta_1) = \{(t^1, t^6)/I(12)\}, \\
\mathcal{M}(\Delta_2) = \{(t^2 + pt^3, t^7)/I(12) | p \in k\}.
\]
These are the components of \( \mathcal{M}_2 \) (see (8)). In this case, \( \mathcal{M}_2 \) is embedded in \( \mathbb{P}^{23} \) by \( \psi \). Let \( J_1 \) (resp. \( J_2 \)) be an elements of \( \mathcal{M}(\Delta_1) \) (resp. \( \mathcal{M}(\Delta_2) \)). Then their represent matrices are
\[
M_{J_1} = 
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]
\[
M_{J_2} = 
\begin{pmatrix}
0 & 0 & 1 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]
respectively. The images \( \psi(J_1) \) and \( \psi(J_2) \) are given in \( \mathbb{P}^{23} \) by the following Plücker coordinates:
\[
\psi(J_1) : \pi_\Lambda = \begin{cases}
1 & \text{for } \Lambda_2 := \{4, 7, 8, 9, 11, 12\}, \\
0 & \text{for the others},
\end{cases}
\]
\[
\psi(J_2) : \pi_\Lambda = \begin{cases}
1 & \text{for } \Lambda_1 := \{3, 7, 8, 9, 11, 12\}, \\
q & \text{for } \Lambda_2 = \{4, 7, 8, 9, 11, 12\}, \\
0 & \text{for the others}.
\end{cases}
\]
Namely, they are represented by the vectors \( \psi(J_1) = (0, \cdots, 1, \cdots, 0) \) (\( \pi_2 = 1 \)), \( \psi(J_2) = (0, \cdots, 1, \cdots, q, \cdots, 0) \) (\( \pi_1 = 1, \pi_2 = q \)). Here we rewrite \( \pi_\Lambda \) and \( \pi_\Lambda \) by \( \pi_1 \) and \( \pi_1 \) respectively. We infer from these facts that \( \psi(M_2) \) consists of all vectors \( a = (0, \cdots, p, \cdots, q, \cdots, 0) \) with the conditions \( \pi_1 = p, \pi_2 = q \) and \( pq \neq 0 \). Since \( \psi \) is injective, we identify \( \psi(M_2) \) with \( M_2 \). Then there exists a natural isomorphism \( \iota : \mathbb{P}^1 \rightarrow M_2 \) defined by \( \iota((p : q)) = a \).

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